1 Introduction

Given any topological space X, consider the relation:

$\dashv := \{(A, B) \in \wp(X) \times \wp(X) : \text{every neighborhood of } A \text{ contains a point of } B\}$

Informally, $A \dashv B$ represents the idea that A must "follow" B under continuous maps. It can be thought of a one-sided notion of attachment between pieces (subsets) of a topological space. Thinking in terms of this notion of attachment can be illuminating and insightful into the intuitive nature of other topological concepts. For example, this directly makes the notion of a continuous function as one that does not "tear" a topological space precise, in the sense that if A is "attached" to B, $f[A]$ must be "attached" to $f[B]$ if f is continuous. The formal statement of this (Theorem 5) is proven later. First, other properties of this relation will be shown.

2 Theorems and Corollaries

2.1 Adherent Points

The definition of $\dashv$ is reminiscent of the definition of an adherent point, suggesting a relationship between the two. If $A \dashv B$, one may think of A as an "adherent set" of B. This first theorem provides a further relationship between the two notions:

Theorem 1. *For all $A, B \subseteq X$, $A \dashv B$ if and only if A contains an adherent point of B.*

Proof. Each implication will be proven separately. First, assume that A contains an adherent point of B. Then, by the definition of an adherent point, there exists a point $a \in A$ such that for every neighborhood N of a, there exists a point $n \in N$ that is also in B. Any neighborhood of A is a neighborhood of a, and therefore contains a point $n \in B$. Therefore, $A \dashv B$.

The second implication will be proven by contrapositive. Assume that A does not contain an adherent point of B. Then every $a \in A$ has a neighborhood N_a that contains no point of B. By the definition of a neighborhood (in terms of open sets), there then exists an open set O_a such that $a \in O_a \subseteq N_a$. Since $O_a \subseteq N_a$, O_a must not contain a point of B either. Let

$$O = \bigcup_{a \in A} O_a$$

Then, by the closure of the open set topology under arbitrary union, O is open. Because there is no a such that O_a contains a point of B, O contains no point of B. Because $a \in O_a$ for all $a \in A$, $O \supseteq A$, and therefore $O \supseteq O \supseteq A$. Therefore, O is a neighborhood of A. Therefore, $A \not\dashv B$. ☐

2.2 Open Sets

The next theorem describes the significance of open sets in terms of $\dashv$. From this perspective, open sets are in some sense "isolated" from the other subsets of the topological space. This, along with the definition of continuity presented later, provides insight into the meaning of the usual definition of continuous functions in terms of open sets. It also serves as a lemma for proving that result. In addition, this provides insight into the notion of conectedness, since it follows trivially from this theorem that a topological space is connected if and only if, informally, it cannot be split into two "unattached" pieces.

Theorem 2. *Any set $O \subseteq X$ is open if and only if $O \not\dashv O^{\mathsf{C}}$.*

Proof. We will again prove each implication separately. For any open set $O \in X$, $O \subseteq O \subseteq O$, so O is a neighborhood of itself. By the definition of a complement, O contains no point of O^{C}. Therefore, $O \not\dashv O^{\mathsf{C}}$.

For any set $O \subseteq X$ such that $O \not\dashv O^{\mathsf{C}}$, there exists a neighborhood U on O that contains no point of O^{C}. By the definition of a neighborhood, there exists an open set V such that $U \supseteq V \supseteq O$, and therefore $U \supseteq O$. Every point in V is also in U, and therefore not in O^{C}. Every point $x \in X$ that is not in O^{C} must be in O, so every point in V must be in O, or $V \subseteq O$. Therefore, $O = V$, and so O is open. □

2.3 Supersets

The first two corollaries are statements about how $\dashv$ behaves with respect to supersets. Informally, these state that if a piece A of X is "attached" to another piece B, then adding more of X to A or B should result in pieces that are similarly attached.

Corollary 1. *For all $A, B \subseteq X$ such that $A \dashv B$ and for any superset C of B, $A \dashv C$.*

Proof. Assume $A, B \subseteq X$ satisfy $A \dashv B$. Then every neighborhood N of A contains a point $n \in B$. For any superset C of B, $n \in C$ by the definition of a superset. Therefore, $A \dashv C$. □

Corollary 2. *For all $A, B \subseteq X$ such that $A \dashv B$ and for any superset C of A, $C \dashv B$.*

Proof. Assume $A, B \subseteq X$ satisfy $A \dashv B$. Then every neighborhood of A contains a point in B. For any superset C of A and any neighborhood N of C, there exists an open set O such that $N \supseteq O \supseteq C$ and therefore $N \supseteq O \supseteq A$. N is then a neighborhood of A, and therefore contains a point in B. Therefore, $C \dashv B$. □

2.4 The Empty Set

The next two corollaries are trivial statements about how attachment behaves with respect to the empty set. Informally, no piece of a topological space is attached to "nothing," and vice versa. They can also be shown using Theorem 2 along with Corollaries 1 and 2, respectively.

Corollary 3. *There exists no $A \subseteq X$ such that $A \dashv \varnothing$.*

Proof. By the definition of an open set topology, X is an open set. Therefore, for any $A \subseteq X$, $X \subseteq X \subseteq A$, so X is a neighborhood of A. By the definition of $\varnothing$, it has no elements, and therefore X contains no element of $\varnothing$. Therefore, $A \not\dashv \varnothing$. □

Corollary 4. *There exists no $A \subseteq X$ such that $\varnothing \dashv A$.*

Proof. $\varnothing = \varnothing$, and so $\varnothing \supseteq \varnothing$. By the definition of an open set topology, $\varnothing$ is open. $\varnothing$ is therefore a neighborhood of itself. Once again, $\varnothing$ has no elements. For any set $A \subseteq X$, $\varnothing$ therefore contains no elements of A, so $\varnothing \not\dashv A$. □

2.5 Neighborhoods

The following theorems describe how neighborhoods can be described in terms of $\dashv$. They provide a different perspective on the behavior of neighborhoods. In many cases, one may think of a neighborhood of a point $x \in X$ as providing a "cushion" of sorts around x, separating it from the outside world. In other cases, where $\{x\}$, no such "cushion" is needed, since x can be thought of as being completely separated (or unattached) from all other points in X.

Theorem 3. *For any point $x \in X$, any set N containing x is a neighborhood of x if and only if $\{x\} \not\dashv N^{\mathsf{C}}$*

Proof. Each implication will be proven separately. First, assume that N is a neighborhood of x. Then there exists an open set O such that $N \supseteq O \ni x$. By Theorem 2, $O \not\dashv O^{\mathsf{C}}$. For any point $n \in N^{\mathsf{C}}$, $n \notin O$, and so $n \in O^{\mathsf{C}}$. Therefore, $N^{\mathsf{C}} \subseteq O^{\mathsf{C}}$. By the contrapositive of Corollary 1, $O \not\dashv N^{\mathsf{C}}$. Because $x \in O$, $\{x\} \subseteq O$. By the contrapositive of Corollary 2, $\{x\} \not\dashv N^{\mathsf{C}}$.

For the second implication, assume that $\{x\} \not\dashv N^{\mathsf{C}}$. Then there exists a neighborhood M of $\{x\}$ containing no point of N^{C}. This implies that every point in M is also in N, or $M \subseteq N$. By the definition of a neighborhood, there exists an open set O such that $N \supseteq O \supseteq \{x\}$. Trivially, $O \ni x$. Therefore, N is a neighborhood of x. □

Theorem 4. *For any set $A \in X$, any set N containing A is a neighborhood of A if and only if $A \not\dashv N^{\mathsf{C}}$*

Proof. Each implication will be proven separately. First, assume that N is a neighborhood of A. Then there exists an open set O such that $N \supseteq O \supseteq A$. By Theorem 2, $O \not\dashv O^{\mathsf{C}}$. For any point $n \in N^{\mathsf{C}}$, $n \notin O$, and so $n \in O^{\mathsf{C}}$. Therefore,

$N^{\complement} \subseteq O^{\complement}$. By the contrapositive of Corollary 1, $O \not\dashv N^{\complement}$. By the contrapositive of Corollary 2, $A \not\dashv N^{\complement}$.

For the second implication, assume that $A \not\dashv N^{\complement}$. Then there exists a neighborhood M of A containing no point of $N^{\complement}$. This implies that every point in M is also in N, or $M \subseteq N$. Therefore, N is a neighborhood of A.

□

2.6 Continuity

This theorem is the motivating property of $\dashv$, formalizing the notion that if a function f is continuous, then the pieces of a topological space that are attached to each other must remain attached after f is applied to the space. A trivial consequence of this is, informatlly, that a bijection f is a homeomorphism if and only if the pieces of the space that are attached before f is applied are precisely those that are attached after f is applied. Naturally, one can think of a topology as a definition of which pieces of a set are attached to each other, a much more intuitive notion than that of an open set topology or even a neighborhood topology.

Theorem 5. *A function $f : X \to Y$ is continuous if and only if for all $A, B \subseteq X$ such that $A \dashv B$, $f[a] \dashv f[b]$.*

Proof. Like before, the two implications will be proven separately. First, assume that $f : X \to Y$ is continuous. Then, for any $A, B \subseteq X$ such that $A \dashv B$ and any neighborhood N of $f[A]$, there exists an open set $O \subseteq Y$ such that $N \supseteq O \supseteq f[A]$. The continuity of f implies that $f^{-1}[O]$ is also open. For every point $a \in A$, $f(a) \in f[A]$ and therefore $f(a) \in O$, so $a \in f^{-1}[O]$. $f^{-1}[O] \supseteq f^{-1}[O] \supseteq A$, so $f^{-1}[O]$ is a neighborhood of A. Because $A \dashv B$, there exists a point $b \in f^{-1}[O]$ that is also in B. By the definition of a preimage and image, $f(b) \in O, f[B]$, respectively. Therefore, $f(b) \in N$, and so $A \dashv B$.

The second implication will again be proven by contrapositive. Assume that $f : X \to Y$ is not continuous. Then there exists an open set $O \subseteq Y$ such that A is not open. For brevity and clarity, let $A \coloneqq f^{-1}[O]$. By Theorem 2, $A \dashv A^{\complement}$. For every $x \in A^{\complement}$, $x \notin A$ by the definition of a complement, and therefore $f(x) \notin O$. This implies that $f(x) \in O^{\complement}$. Therefore, $f[A^{\complement}] \subseteq O^{\complement}$. Again by Theorem 2, $O \not\dashv O^{\complement}$. By the contrapositive of Corollary 1, $O \not\dashv f[A^{\complement}]$. For all $a \in A$, $f(a) \in O$, so $f[A] \subseteq O$. By the contrapositive of Corollary 2, $f[A] \not\dashv f[A^{\complement}]$. □